Facility Location and Strategic Supply Chain Management

Structure

I. Location Concepts

II. Location Theory
   Chapter 3 – Continuous Location Problems
   Chapter 4 – Location Problems on Networks
   Chapter 5 – Discrete Location Problems
   Chapter 6 - Territory Design

III. Strategic Supply Chain Management
Location Theory

Main task

Locate one or more than one new facilities in relation to already existing facilities.

Relation between already existing and new facilities

is normally expressed by the necessity that existing facilities satisfy their demands by and at the new facilities, respectively (depending on whether the customer goes to the facility or „the facility to the customer“).

Simplification

From now on, existing facilities are called customers.
Location Theory

We will consider three classes of location problems:

Continuous Location Problems

- New facilities can be located anywhere in the plane. Plane: typically a part of a map.
- The most simple, but also the most unrealistic class of problems.
- Many real world requirements cannot be considered, or it’s very difficult to include them.
- Are still applied nowadays in practice in spite of or perhaps because of their simplicity.
Location Theory

Location Problems on Networks

- New facilities can be located only on a network, e.g. road or railway networks.
- These approaches are more real world-oriented than planar location problems.

Discrete Location Problems

- New facilities can be located only in a finite set of given locations, e.g. on previously selected properties.
- Most realistic and most flexible class of location problems by far.
- Constraints, demands and guidelines from practice can be considered almost unrestrictedly in the modeling.
Location Theory

Further distinctive features

Number of new facilities

Methodologic and over all algorithmic differentiation whether one or more than one new facilities are to be located at once.

„Effects“ of satisfying the customer demands

Typical forms of these „effects“

- Costs which occur by satisfying the customer demand by and at the existing facilities, respectively.
  Most often a large part of these costs are transportation costs → are therefore highly dependent on the distance between facilities.

- Profits which are achieved by satisfying the customer demands.
Location Theory

Objective function(s)

Have an important influence on the actual specification of the problem.

Examples for objective functions:

- Median / Sum – function
  The „effects“ which are involved by the satisfaction of the customer demands are added up.

Example:
Location of factories, warehouses or retail agencies where the sum of the occurring (transportation-) costs or achieved profits are to be minimized or maximized.
Location Theory

- **Center / Maximum – function**
  
  Calculates among all particular „effects“, that involve the **satisfaction** of a **customer demand**, the **maximal one**.

  **Example:**
  Placement of a public facility, e.g. schools or fire departments where the **maximal distance** from a customer to the location of the new facility is to be **minimized**.
Location Theory

Furthermore distinguish
• cost-orientated and \( \rightarrow \) Minimization
• profit-orientated objective functions \( \rightarrow \) Maximization

Since, regarding the latter functions, the costs are very important too, the differentiation is a little ambiguous.

Distinguish between objective functions with
• attractive or
• repulsive character

**Attractive** and **repulsive**, respectively, means that the customers in the latter case want to be as far away as possible from the new facility (e.g. a waste incineration plant, power station), in the former case however as near as possible to the new facility (e.g. a cinema or department store).

**Multiple objective functions** \( \rightarrow \) **multi-criteria** (location) problem.
Location Theory

Chapter 3 – Continuous Location Problems

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• 1 – Median Problems

• 1 – Center Problems

• Multi-Facility Problems
Chapter 3 – Continuous Location Problems

Introduction

Topic of the chapter

The cost-orientated location of one or several new facilities in the plane, $\mathbb{IR}^2$.

Make the following simplifying assumptions

Locations
- are idealized as points, i.e. have no spatial dimension

New facilities $X = \{x_1, \ldots, x_p\}$, $p \geq 1$
- can be located anywhere on the plane
- their locations are represented by coordinates $x_i = (x_{i,1}, x_{i,2}) \in \mathbb{IR}^2$ $\forall i = 1, \ldots, p$
- Note: for just one new location $x = (x_1, x_2) \in \mathbb{IR}^2$
Introduction

**Existing facilities (customers)** $A = \{a_1, \ldots, a_n\}$

- are represented by a set of points on the plane

$$a_i = (a_{i,1}, a_{i,2}) \in \mathbb{R}^2 \quad \forall i = 1, \ldots, n$$

Note: if just one dimension $a_i = a_{i,1} \in \mathbb{R} \quad \forall i = 1, \ldots, n$

- a **positive weight** $w_i > 0$ is assigned to every customer $a_i \in A$  
  Example
  - customer's demand for a certain product
  - number of inhabitants at this location
  - costs per **distance unit** for satisfying the customer demand

**Costs**

- Only **transportation costs** are considered.
  These are proportional to the quantity and to the covered distance.

- No further variable or fixed costs.
Introduction

The Fermat-Weber Problem

First formulations of planar location problems

- **Fermat** (1601 – 1665)
- **Torricelli** (1608 – 1647)

First papers about economic location problems on the plane

- **Launhardt** (1887)
- **Alfred Weber** (1909)

Fermat’s and Torricelli’s problem

- Let **three points**, \( a_1, a_2, a_3 \in \mathbb{R}^2 \), be on the plane with identical weights \( w_1 = w_2 = w_3 = 1 \).

- **Euclidean distance** as measure for the distance between 2 points \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \):

\[
d_{\text{Euklid}}(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}
\]
The Fermat-Weber Problem

Formal description of the problem

A point \( x = (x_1, x_2) \) which minimizes the sum of the distances from this point to the three already existing points.

\[
\min_{x \in \mathbb{R}^2} \sum_{i=1}^{3} d_{\text{Euklid}}(x, a_i)
\]

First solution method

Torricelli’s Geometrical construction
The Fermat-Weber Problem

Toriccelli‘s Method

1. Connect the three points $a_1$, $a_2$ and $a_3$ to get a triangle.

2. Construct on each of the sides of the triangle an equilateral triangle whose third point is at the outside.

3. Construct for each of these equilateral triangles the circumcircle.

4. The intersection point of these three circumcircles is the optimal location, the so called Torricelli point.
Introduction

Distance Measures

Calculating the transportation costs between two locations can be reduced to the computation of the distance between the two points.

Metric concepts are used in order to compute distances in planar location problems.

A mapping \(d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) is called a metric or distance function if the following holds for all \(x, y, z \in \mathbb{R}^n, n \geq 1\),

1. \(d(x, y) \geq 0, \quad d(x, y) = 0 \iff x = y\) (positive definiteness)
2. \(d(x, y) = d(y, x)\) (symmetry)
3. \(d(x, y) \leq d(x, z) + d(z, y)\) (triangle inequality)
Distance Measures

$l_p$ – metric as a distance measure for two points $x, y \in \mathbb{R}^2$ on the plane:

$$l_p(x, y) = \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p}, \quad p > 0$$

Rectangular distance $l_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$

Also called $l_1$ or Manhattan Metric. Very important in in-house location planning as well as in facility location planning in cities.

Represents distances between points in areas where one can only move horizontally or vertically.

Example
- in warehouses whose room layout is rectangular
- in American cities

$$l_1(x, y) = |2 - 4| + |4 - 2| = 4$$
Distance Measures

**Euclidean distance**

\[ l_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \]

Also called \( l_2 \)– metric. Can be used to estimate real road distances.

Example

- Planning power supply lines or pipeline-systems

**Squared Euclidean distance**

\[ l_2^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \]

Farther points are more important.

Example

- **Locations for fire department stations**
  not only the sum of all distances is important, but also the maximum length of a trip.

\[ l_2^2(x, y) = (2 - 4)^2 + (4 - 2)^2 = 8 \]
Location Theory

Chapter 3 – Continuous Location Problems

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Chapter 3 – Continuous Location Problems

1 – Median problems

Task

Locate a new facility on the plane such that the sum of all weighted distances between the new location and the existing facilities is minimal.

Formula

\[
\min_{x \in \mathbb{R}^2} f(x) := \sum_{i=1}^{n} w_i d(x, a_i)
\]

where

- \( A = \{a_1, \ldots, a_n\} \) denote the \( n \) existing facilities
- \( w_i > 0, i = 1, \ldots, n \), denote their weights
- \( x \) denotes the location of the new facility and
- \( d \) is a metric.
A point \( x^* \in \mathbb{R}^2 \) that minimizes the function \( f(x) \), i.e.
\[
f(x^*) \leq f(x), \ \forall x \in \mathbb{R}^2,
\]
is called optimal. Let the set of all optimal points of the function \( f(\cdot) \) be \( \mathcal{X}^*(f) \).

**Analytic characteristics of 1 – median problems with \( l_p \) – metric**

The median objective function \( f(x) \) is convex
\[ \rightarrow \ \text{every local minimum is also a global one} \]

Furthermore,
there is always at least one optimal solution in the convex hull of the customer locations
\[
\mathcal{X}^*(f) \cap \text{Conv}(A) \neq \emptyset
\]
1 – Median Problems

Dominance criterion

Let $d$ be a $l_p$-metric. If for a customer $a_k$ with weight $w_k$ holds that

$$w_k \geq \frac{1}{2} \sum_{i=1}^{n} w_i$$

i.e. the customer has at least half of the total weights, then the location of customer $a_k$ is an optimal solution of the problem.
1 – Median Problems

The 1 – Median Problem with $l_1$ - Metric

Objective function of the 1 – Median Problem with $l_1$ – Metric

$$f(x) = \sum_{i=1}^{n} w_i l_1(x, a_i)$$

$$= \sum_{i=1}^{n} w_i (|x_1 - a_{i,1}| + |x_2 - a_{i,2}|)$$

$$= \sum_{i=1}^{n} w_i |x_1 - a_{i,1}| + \sum_{i=1}^{n} w_i |x_2 - a_{i,2}|$$

Since $x_1$ appears just in the first sum and $x_2$ only in the second one, the objective function can be separated into two independent functions

$$f_1(x_1) := \sum_{i=1}^{n} w_i |x_1 - a_{i,1}| \quad \text{and} \quad f_2(x_2) := \sum_{i=1}^{n} w_i |x_2 - a_{i,2}|$$
1 – Median Problems with $l_1$ - Metric

**Minimization** of the function $f(x)$

Determine separately the minimum of $f_1(x)$ and $f_2(x)$.

We obtain the optimal solution $\mathcal{X}^*(f)$ of the previous problem arises by combining the single solutions $\mathcal{X}^*(f_1)$ and $\mathcal{X}^*(f_2)$

$\Rightarrow$ the initial task reduces to two 1 – dimensional problems, so called „problems on the line“.

The 1 – Median Problem with $l_1$ – Metric on the Line

The following problem has to be solved

$$\min_{x \in \mathbb{R}} f(x) := \sum_{i=1}^{n} w_i |x - a_i|$$

where

- $A = \{a_1, \ldots, a_n\}, a_i \in \mathbb{R} \ \forall i$, denote the $n$ existing facilities (on the line) and
- $x \in \mathbb{R}$ denotes the location of the new facility.
1 – Median Problems with $l_1$ – Metric on the Line

Algorithm

1. Compute the sum $W$ of all weights $W := \sum_{i=1}^{n} w_i$

2. If the dominance criterion is satisfied for a customer $a_k$, i.e. $w_k \geq \frac{1}{2} W$, then the customer location $a_k$ is an optimal solution, $x^* = a_k$
   \[ \rightarrow \text{Stop} \]

3. Sort the customer locations $a_1, \ldots, a_n$ by non-decreasing coordinates
   \[ a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_n} \]

4. Sort the weights analogously depending on the customer locations and compute the index $h$ for which
   \[ \sum_{j=1}^{h-1} w_{i_j} < \frac{1}{2} W \quad \text{and} \quad \sum_{j=1}^{h} w_{i_j} \geq \frac{1}{2} W. \]

5. $x^* = a_{i_h}$ denotes the coordinate of an optimal location, $\mathcal{X}^*(f) = \{a_{i_h}\}$. 
1 – Median Problems with $l_1$-Metric

Remarks

• At least one **optimal solution** of the problem can always be found in the set of customer locations.

• **Step 4.** of the algorithm can effectively be solved in the following way: the **weights**, beginning with the first one, are **added step by step** until this **sum** just contains at least **half** of the **total weights**.

• If the sum of these weights is exactly equal to the half of the total, i.e.

$$\sum_{j=1}^{h} w_{ij} = \frac{1}{2} W,$$

then the **complete interval** $[a_{i_h}, a_{i_{h+1}}]$ is **optimal**: $\mathcal{X}^*(f) = [a_{i_h}, a_{i_{h+1}}]$.

• Complexity of the algorithm: $O(n \cdot \log n)$. 
1 – Median Problems with $l_1$ - Metric

Remarks (cont.)

• We can formulate the 1 – median problem with $l_1$ - metric as a linear program

$$\min_{x \in \mathbb{R}^2} \sum_{i=1}^{n} w_i l_1(x, a_i) \iff \min \sum_{i=1}^{n} w_i z_i$$

s.t. \( |x_1 - a_{i,1}| + |x_2 - a_{i,2}| \leq z_i \) for all \( i = 1, \ldots, n \)

\( z_i \geq 0 \) for all \( i = 1, \ldots, n \)

$$\iff \min \sum_{i=1}^{n} w_i z_i$$

s.t. \( x_1 + x_2 - a_{i,1} - a_{i,2} \leq z_i \) for all \( i = 1, \ldots, n \)

\(-x_1 + x_2 + a_{i,1} - a_{i,2} \leq z_i \) for all \( i = 1, \ldots, n \)

\(-x_1 - x_2 + a_{i,1} + a_{i,2} \leq z_i \) for all \( i = 1, \ldots, n \)

\( x_1 - x_2 - a_{i,1} + a_{i,2} \leq z_i \) for all \( i = 1, \ldots, n \)

\( z_i \geq 0 \) for all \( i = 1, \ldots, n \)
1 – Median Problems with $l_1$ - Metric

Example: 2 – dimensional problem

Let $A = \{(1,4), (2,6), (5,1), (4,2), (6,5)\}$ and $w = \{2,3,1,1,2\}$. Customer locations for the first subproblem

$$a_{1,1} = 1, a_{2,1} = 2, a_{3,1} = 5, a_{4,1} = 4, a_{5,1} = 6$$

Algorithm

1. $W = 2 + 3 + 1 + 1 + 2 = 9$.
2. Dominance criterion is not satisfied for any customer.
3. Sorted customer locations: $a_{1,1} \leq a_{2,1} \leq a_{4,1} \leq a_{3,1} \leq a_{5,1}$
4. As a consequence of Step 3. $\Rightarrow$ Order of the weights that have to be added:

$$\{w_1, w_2, w_4, w_3, w_5\} = \{2, 3, 1, 1, 2\}.$$  

$\Rightarrow w_1 = 2 < \frac{1}{2}W = 4.5$ and $w_1 + w_2 = 5 \geq \frac{1}{2}W$.

5. $x_1 = a_{2,1} = 2$ is the optimal location for the first subproblem.
Example

Customer locations for the second subproblem

\[ a_{1,2} = 4, \ a_{2,2} = 6, \ a_{3,2} = 1, \ a_{4,2} = 2, \ a_{5,2} = 5 \]

Steps 1. and 2. remain the same for the second subproblem.

Method

3. Sorted customer locations: \[ a_{3,2} \leq a_{4,2} \leq a_{1,2} \leq a_{5,2} \leq a_{2,2} \]

4. As a consequence of step 3. ⇒ Order of the weights that have to be added
   \[ \{w_1, w_2, w_4, w_3, w_5\} = \{1, 1, 2, 2, 3\} \]
   ⇒ \[ w_3 + w_4 + w_1 = 4 < \frac{1}{2}W \] and \[ w_3 + w_4 + w_1 + w_5 = 6 \geq \frac{1}{2}W. \]

5. So, \[ x_1 = a_{5,2} = 5 \] is the optimal location for the second subproblem.

Optimal solution of the problem \( x^* = (2, 5) \)

with objective function value \( f(x^*) = 27. \)
1 – Median Problems with $l_1$-Metric

**Geometric interpretation**

If one draws a *horizontal* and a *vertical line* trough each of the customer locations, then these lines form a *rectangular grid*. Among the *intersection points* of these *lines* one can always find an *optimal solution*. The intersection points form a so called *finite dominating set (FDS)* for the problem.

![Geometric Interpretation Diagram]

$\mathbf{x}^* = (2, 5)$
Geometric Interpretation

If it's not possible to locate a facility directly at the optimal location, e.g. because there is already a machine there, then one is interested in alternative solutions that have minimal costs regarding the given constraints for the location.

Therefore it is useful to compute so called level curves, i.e. sets of points that have the same costs. In case of the $L_1$ – metric these level curves consist of line segments within the grid.
1 – Median Problems with $l_1$-Metric

Restrictive Location Problems

Forbidden area $R$

**Convex subset** of the plane where the placement of new facilities is not allowed.

If

\[ X^*(f) \subset \text{int}(R) \]

holds i.e. the optimal solution set of the problem lies completely within the forbidden area, then the **intersection points** of the **horizontal** and **vertical lines** through the customer locations with the boundary of $R$ form an FDS for the problem.
1 – Median Problems

The 1 – Median Problem with $l_2^2$ - Metric

**Objective function** of the 1 – Median Problem with $l_2^2$ – Metric

\[
\begin{align*}
    f(x) &= \sum_{i=1}^{n} w_i l_2^2(x, a_i) \\
    &= \sum_{i=1}^{n} w_i ((x_1 - a_{i,1})^2 + (x_2 - a_{i,2})^2) \\
    &= \sum_{i=1}^{n} w_i (x_1 - a_{i,1})^2 + \sum_{i=1}^{n} w_i (x_2 - a_{i,2})^2 \\
    &=: f_1(x_1) + f_2(x_2)
\end{align*}
\]

⇒ The objective function can be separated again into two independent terms.

Note: $f_1(x_1)$ and $f_2(x_2)$ are **differentiable**
1 – Median Problems with $l_2^2$-Metric

Minimization of $f_1(x_1)$ and $f_2(x_2)$

Differentiate with respect to $x_1$ and $x_2$, respectively, and set the particular derivative to zero

$$0 = \partial f_k(x_k) = \sum_{i=1}^{n} w_i \partial[(x_k - a_{i,k})^2] = \sum_{i=1}^{n} 2 w_i (x_k - a_{i,k}), \quad k = 1, 2$$

Solve the equations for $x_1$ and $x_2$, respectively:

$$x_k = \frac{\sum_{i=1}^{n} w_i a_{i,k}}{\sum_{i=1}^{n} w_i}, \quad k = 1, 2$$

Optimal solution for the 1 – median problem with $l_2^2$-metric

$$x^* = \left( \frac{\sum_{i=1}^{n} w_i a_{i,1}}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} w_i a_{i,2}}{\sum_{i=1}^{n} w_i} \right).$$

$x^*$ is also called centroid of the locations $a_1, \ldots, a_n$. 

Facility Location and Strategic Supply Chain Management
Prof. Dr. Stefan Nickel
1 – Median Problems with $l_2^2$-Metric

Example

Consider again $A = \{(1,4), (2,6), (5,1), (4,2), (6,5)\}$ and $w = \{2,3,1,1,2\}$.

As centroid one gets

\[
x_1 = \frac{\sum_{i=1}^{n} w_i a_{i,1}}{\sum_{i=1}^{n} w_i} = \frac{29}{9} = 3.22
\]

\[
x_2 = \frac{\sum_{i=1}^{n} w_i a_{i,2}}{\sum_{i=1}^{n} w_i} = \frac{13}{3} = 4.33
\]

The level curves are circles.
1 – Median Problems

The 1 – Median Problem with $l_2$ – Metric

Objective function of the 1 – Median Problem with $l_2$ – Metric

$$f(x) = \sum_{i=1}^{n} w_i l_2(x, a_i) = \sum_{i=1}^{n} w_i \sqrt{(x_1 - a_{i,1})^2 + (x_2 - a_{i,2})^2}$$

Problem

• The objective function for the $l_2$ – metric cannot be separated and
• is not differentiable at the existent customer locations $a_i$

For $x \neq a_i$, for all $i$, holds

$$\frac{\partial f(x)}{\partial x_k} = \sum_{i=1}^{n} w_i \frac{\partial l_2(x, a_i)}{\partial x_k} = \sum_{i=1}^{n} w_i \frac{2 (x_k - a_{i,k})}{2 \sqrt{(x_1 - a_{i,1})^2 + (x_2 - a_{i,2})^2}}$$

$$= \sum_{i=1}^{n} w_i \frac{(x_k - a_{i,k})}{l_2(x, a_i)} \quad k = 1, 2$$
1 – Median Problems with $l_2$-Metric

**Algorithm to solve the problem**

1. Look for an optimal solution in the set of customer locations.

2. If there is no optimal solution in the set of customer locations, set the partial derivatives to zero and solve for $x_1$ and $x_2$, respectively.

**To 1.**

Strong dominance criterion for the Euclidean distance.

The location $a_j$ of a customer is optimal if

$$
\gamma(a_j) := l_2 \left( \sum_{i=1}^{n} w_i \frac{a_j - a_i}{l_2(a_j, a_i)}, 0 \right) \leq w_j
$$
1 – Median Problems with $l_2$-Metric

To 2.

Setting the partial derivatives to zero and solving results in

$$\partial f(x) = 0 \quad \Leftrightarrow \quad x = \frac{\sum_{i=1}^{n} w_i \cdot \frac{a_i}{l_2(x, a_i)}}{\sum_{i=1}^{n} w_i}$$

In this term, we cannot isolate $x$!

Weiszfeld (1937): Approximate the optimal solution using an iterative scheme

Start with any given values $(x_1^{(0)}, x_2^{(0)})$ and insert these values into the right hand side of the equation above. This yields new coordinates $(x_1^{(1)}, x_2^{(1)})$ on the left hand side. Inserting these new values again into the right hand side. We obtain new coordinates $(x_1^{(2)}, x_2^{(2)})$, etc.

Stop the method once a given stopping criterion is satisfied, e.g. if the difference between two consecutive coordinates or objective function values is lower than a certain threshold.
1 – Median Problems with $l_2$-Metric

Weiszfeld’s Approximation Method

1. Using the strong dominance criterion, check if one of the existing locations $a_j$ is already an optimal solution for the problem, i.e. $\gamma(a_j) \leq w_j$. If so, then set $x^* = a_j$ and stop the method.

2. Let $\ell := 0$ and initialize the starting point $x^{(0)}$ of the iteration with the centroid

$$x^{(0)} := \frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}$$

3. Set

$$x^{(\ell+1)} := \frac{\sum_{i=1}^{n} w_i \frac{a_i}{l_2(x^{(\ell)}, a_i)}}{\sum_{i=1}^{n} w_i \frac{1}{l_2(x^{(\ell)}, a_i)}}$$

4. If the relative difference between two consecutive objective function values is less then a value $\delta$

$$\delta^{(\ell+1)} := \frac{f(x^{(\ell)}) - f(x^{(\ell+1)})}{f(x^{(\ell)})} \leq \delta$$

then stop the algorithm. Otherwise set $\ell := \ell + 1$ and continue with step 3.
1 – Median Problems with $l_2$-Metric

Example

Let again $A = \{(1,4), (2,6), (5,1), (4,2), (6,5)\}$ and $w = \{2,3,1,1,2\}$.

Weiszfeld – Algorithm

1. The strong dominance criterion is not satisfied for any customer location

   $\gamma(a_1) = 5.3 > 2 = w_1$, $\gamma(a_2) = 4.5 > 3 = w_2$, $\gamma(a_3) = 7.2 > 1 = w_3$,
   
   $\gamma(a_4) = 4.9 > 1 = w_4$, $\gamma(a_5) = 5.8 > 2 = w_5$

2. Set $x^{(0)} = (3.22, 4.33)$ as the centroid.

3. One gets the following iterations

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$x_1^{(\ell)}$</th>
<th>$x_2^{(\ell)}$</th>
<th>$f(x^{(\ell)})$</th>
<th>$\delta^{(\ell+1)}$</th>
</tr>
</thead>
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<td>1</td>
<td>3.222</td>
<td>4.333</td>
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<td>-</td>
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<td>4.799</td>
<td>22.039</td>
<td>0.001</td>
</tr>
</tbody>
</table>

4. For $\delta = 0.001$ the algorithm stops after the 5th iteration.
Example

Now the level curves are not that well-shaped any more.
Remarks

- The Weiszfeld Algorithm is a descent method, i.e. the objective function value gets strictly smaller in each iteration

\[ f(x^{(\ell+1)}) < f(x^{(\ell)}). \]

- If the customer locations are not all on a line, then there is an unique optimal solution.

- If none of the points \( x^{(\ell)} \) coincides with a customer location, then the Weiszfeld algorithm converges towards the optimal solution of the problem.

- If one of the points \( x^{(\ell)} \) is at a customer location, then the term in step 3 is not well defined (the denominator is 0).

  To avoid this problem the so called Hyperboloid Approximation Method was introduced. There is just a little difference to the Weiszfeld Algorithm in the form of the objective function

\[ f(x) = \sum_{i=1}^{n} w_i \sqrt{(x_1 - a_{i,1})^2 + (x_2 - a_{i,2})^2 + \varepsilon}, \quad \varepsilon > 0. \]

  The value of \( \varepsilon \) lies typically in the interval from \( 10^{-3} – 10^{-6} \).
1 – Median Problems with $l_2$-Metric

Varignon frame

The 1 – median problem with $l_2$ – metric can also be solved in a mechanical way.

**Steps**

- **Project** the $n$ customer locations true to scale on a plate and **drill** a hole at each of the projection points.

- **Tie up** the ends of $n$ threads in a knot and put the other ends through the drilled holes. Attach to each loose end of a thread a weight equal to the demand $w_i$ of the node corresponding to that hole.

- In the balance of forces the knot gives the optimal location.
Location Theory

Chapter 3 – Continuous Location Problems

Contents

• Introduction

• 1 – Median Problems

• 1 – Center Problems

• Multi-Facility Problems
Chapter 3 – Continuous Location Problems

1 – Center Problems

Locate a new facility in the plane such that the maximal weighted distance from the new location to the already existing facilities becomes minimal.

Formal

\[ \min_{x \in \mathbb{R}^2} g(x) \text{ with } g(x) := \max_{i=1,\ldots,n} w_i d(x, a_i) \]

where

- \( A = \{a_1, \ldots, a_n\} \) denote the \( n \) existing facilities
- \( w_i > 0, i = 1, \ldots, n \), denote the corresponding weights,
- \( x \) denotes the location of the new facility and
- \( d \) is a metric.

A point \( x^* \in \mathbb{R}^2 \) is called optimal if it minimizes the function \( g(x) \), i.e. \( g(x^*) \leq g(x), \forall x \in \mathbb{R}^2 \).
1 – Center Problems

Let the set of all optimal points of the function $g(\cdot)$ be $X^*(g)$.

A problem is called unweighted, if all weights are equal to one

$w_i = 1, \ \forall i, \ldots, n.$

**Remark:**

If $w_i = w, \ \forall i, \ldots, n$, with $w > 0$, then this corresponds to an unweighted problem.

**Properties of 1 – center problems with $l_p$ - metric**

The center objective function $g(x)$ is convex

$\rightarrow$ every local minimum is a global one

Furthermore,

one can always find an optimal solution in the convex hull of the
customer locations $X^*(g) \cap Conv(A) \neq \emptyset$. 
1 – Center Problems

Applications

Location of

- **public facilities**, e.g. schools, libraries, …
- **emergency facilities**, e.g. ambulance depots, fire department stations, helicopter bases, …
- **mobile phone masts, radio masts**, …
1 – Center Problems

The 1 – Center Problem with $l_1$ - Metric

**Objective function** of the 1 – Center Problem with $l_1$ – Metric

$$g(x) = \max_{i=1,\ldots,n} w_i l_1(x, a_i)$$

$$= \max_{i=1,\ldots,n} w_i (|x_1 - a_{i,1}| + |x_2 - a_{i,2}|)$$

To simplify the problem we **transform the coordinates** of all customer locations.

Assume that

- $T$ is the **transformation matrix** $T := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
- $x'$ is a **transformed** point:

$$x' = (x'_1, x'_2) := (x_1, x_2) \cdot T = (x_1, x_2) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = (x_1 + x_2, -x_1 + x_2)$$
1 – Center Problems

**Inverse transformation** of a point\(^t\):

\[ x = x' \cdot T^{-1} = (x_1', x_2') \]
\[ = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right) \]
\[ \frac{1}{2}(x'_1 - x'_2, x'_1 + x'_2) \]

Example: \( x = (2, 3) \)

\[ x' = x \cdot T = (2, 3) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) = (2+3, -2+3) = (5, 1) \]

and

\[ x = x' \cdot T^{-1} = (5, 1) \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right) = \frac{1}{2}(5-1, 5+1) = (2, 3) \]

**Geometrically** the transformation corresponds to a clockwise rotation of the points by 45° around the origin and, a subsequent dilation of the coordinates by the factor \( \sqrt{2} \).
1 – Center Problems

Effects of the **transformation** on the **calculation of the distance** between two points \( x \) and \( y \)

\[
l_1(x, y) = |x_1 - y_1| + |x_2 - y_2|
\]

\[
= max\{|x_1 - y_1 + x_2 - y_2|, -(x_1 - y_1) + x_2 - y_2|\}
\]

\[
= max\{|x'_1 - y'_1|, |x'_2 - y'_2|\}
\]

\[
=: l_\infty(x', y')
\]

\( \Rightarrow l_1 \) – distance between \( x \) and \( y \) is **identical** to \( l_\infty \) - distance between \( x' \) and \( y' \).

The metric \( l_\infty \) is also denoted as **Tchebychev – metric**.
1 – Center Problems

Relation between $l_1$ and $l_\infty$

<table>
<thead>
<tr>
<th>$l_1$ – center problem</th>
<th>$\min_{x \in \mathbb{R}^2} \max_{i=1,...,n} w_i l_1(x, a_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_\infty$ – center problem with transformed coordinates</td>
<td>$\min_{x' \in \mathbb{R}^2} \max_{i=1,...,n} w_i l_\infty(x', a'_i)$</td>
</tr>
</tbody>
</table>

Solution Method for the $l_1$ – Center Problem

1. **Transform** all customer locations $A' := \{a'_i\}$ with $a'_i = a_i \cdot T$
2. Find an **optimal solution** $x_{\infty}^*$ of the $l_\infty$ – center problem with the customer locations $A'$.
3. Obtain an optimal solution $x^*$ of the $l_1$ – problem: $x^* = x_{\infty}^* \cdot T^{-1}$

$\Rightarrow$ Find a **solution** for the $l_\infty$ – center problem
1 – Center Problems

The 1 – Center Problem with $l_\infty$ - Metric

Objective function of the 1 – center problem with $l_\infty$ – metric

$$g(x) = \max_{i=1,...,n} w_i l_\infty(x, a_i)$$

$$= \max_{i=1,...,n} w_i \max\{|x_1 - a_{i,1}|, |x_2 - a_{i,2}|\}$$

The unweighted case: The Square-Covering Problem

Reformulation of the problem

$$\min_{x \in \mathbb{R}^2} \max_{i=1,...,n} l_\infty(x, a_i) \iff \min_{\text{u.d.N.}} z \quad \text{such that} \quad l_\infty(x, a_i) \leq z \quad \forall i = 1, \ldots, n$$

$$\Rightarrow \textbf{Minimize } z \text{ such that the distance from a point } x \text{ to all customer locations is smaller then or equal to } z.$$
Unweighted 1 – Center Problems with $l_\infty$ - Metric

Geometric interpretation of the relation

$$l_\infty(x, a_i) \leq z \quad \forall i = 1, \ldots, n \quad (*)$$

The set of all points with $l_\infty$ - distance $z$ to the origin

$$B_\infty(z) = \{ y \in \mathbb{R}^2 : l_\infty(0, y) = z \}$$

yields a square

Defined by the vertices $(z, z)$, $(-z, z)$, $(-z, -z)$ and $(z, -z)$
⇒ If \((*)\) holds for \((x, z)\), then all customer locations are within a square with center point \(x\) and "radius" \(z\).

The Minimal Square-Covering Problem

Find a square with
- minimal "radius" \(z\) and
- center point \(x\),
that covers all customer locations, i.e. the distance from \(x\) to all customers is smaller then or equal to \(z\).

It holds

An optimal solution \((x^*, z^*)\) of the square-covering problem is also an optimal solution of the unweighted \(l_\infty\) – center problem, and vice versa.
Unweighted 1 – Center Problems with $l_\infty$ - Metric

**Algorithm to compute all optimal solutions**

1. Compute the smallest rectangle $R$, that includes all customer locations. $R$ is uniquely defined by two opposite vertices (ul, or)

2. If $R$ is a square, then $x^* = \text{center of } R \rightarrow \text{STOP}$

3. Expand the rectangle $R$ along the shorter side successively in both directions to a square $Q_1$ and $Q_2$.

4. The connecting line between the center points $M_1$ and $M_2$ of the two squares $Q_1$ and $Q_2$ is the set of optimal solutions

$$\mathcal{X}^*(g) = \overline{M_1 M_2}$$

$$= \{ x \in \mathbb{R}^2 : x = \lambda M_1 + (1 - \lambda) M_2, \text{ for all } 0 \leq \lambda \leq 1 \}$$
Unweighted 1 – Center Problems with $l_\infty$ - Metric

Example

Let $A = \{(1,4), (2,6), (5,1), (4,2), (8,5)\}$

Method

1. Circumscribing rectangle $R = (ul, or)$
   
   $$ul = (\min_{i=1,\ldots,n} a_{i,1}, \min_{i=1,\ldots,n} a_{i,2}) = (1, 1)$$
   
   $$or = (\max_{i=1,\ldots,n} a_{i,1}, \max_{i=1,\ldots,n} a_{i,2}) = (8, 6)$$

2. No square $\rightarrow$ 3.

3. Successively expand the rectangle upwards and downwards to a square
   
   $$Q_1 = (ul, (or_1, ul_2 + (or_1 – ul_1))) = ((1,1), (8,8))$$ and
   
   $$Q_2 = ((ul_1, or_2 - (or_1 – ul_1)), or) = ((1,-1), (8,6))$$

4. Center points $M_1 = (4.5, 4.5)$ and $M_2 = (4.5, 2.5)$

   $\Rightarrow X^*(g) = \{4.5\} \times [2.5, 4.5]$
Remark
If we just want to determine one optimal location, then the method is much shorter.

Method to determine an optimal solution
1. Calculate the circumscribing rectangle $R$ for the customer locations. $R$ is definitely determined by two opposite vertices $(ul, or)$.
2. $x^* = center point of R$ is an optimal location.

Example (continued)
Method
1. Circumscribing rectangle $R = (ul, or) = ((1,1), (8,6))$
2. $x^* = center point of ((1,1), (8,6)) = ((1+8)/2, (1+6)/2) = (4.5, 3.5)$.

$x^*$ lies on the connection line between $M_1 = (4.5, 4.5)$ and $M_2 = (4.5, 2.5)$. 
1 – Center Problems with $l_\infty$ - Metric

The general, weighted case

Objective function of the 1 – Center Problem with $l_\infty$ – Metric

$$g(x) = \max_{i=1,\ldots,n} w_i \max\{|x_1 - a_{i,1}|, |x_2 - a_{i,2}|\}$$

$$= \max\left\{ \max_{i=1,\ldots,n} w_i |x_1 - a_{i,1}|, \max_{i=1,\ldots,n} w_i |x_2 - a_{i,2}| \right\}$$

$$g_1(x_1) \quad g_2(x_2)$$

⇒ The $l_\infty$ – center problem can be separated into two independent subproblems.

Minimization of the function $g(x)$

Determine separately the minimum of the terms $g_1(x)$ and $g_2(x)$.

Combination of the singular solutions leads to the optimal solution.

⇒ Original task is reduced to the solving two 1 – dimensional problems on the line.
1 – Center Problems with $l_\infty$ - Metric

The 1 – Center Problem with $l_\infty$ – Metric on the Line

The following problem has to be solved:

$$\min_{x \in \mathbb{R}} g(x) := \max_{i=1,\ldots,n} w_i |x - a_i|$$

where

- $A = \{a_1, \ldots, a_n\}, \ a_i \in \mathbb{R} \ \forall i$, denote the $n$ existing facilities (on the line)
- $w_i > 0, \ i = 1, \ldots, n$, denote the corresponding weights
- $x \in \mathbb{R}$ denotes the location of the new facility.
Reformulation as a linear program

\[
\begin{align*}
\min_{x \in \mathbb{R}} \max_{i=1, \ldots, n} w_i |x - a_i| & \quad \Leftrightarrow \quad \min_{} z \\
\text{s.t.} \quad |x - a_i| & \leq \frac{z}{w_i} \quad \forall i = 1, \ldots, n \\
\end{align*}
\]

\[
\begin{align*}
\leftarrow \begin{array}{c}
\min_{} z \\
\text{s.t.} \quad x - a_i & \leq \frac{z}{w_i} \quad \forall i = 1, \ldots, n \\
-x + a_i & \leq \frac{z}{w_i} \quad \forall i = 1, \ldots, n \\
\end{array} \\
\leftarrow \begin{array}{c}
\min_{} z \\
\text{s.t.} \quad x & \leq a_i + \frac{z}{w_i} \quad \forall i = 1, \ldots, n \\
x & \geq a_i - \frac{z}{w_i} \quad \forall i = 1, \ldots, n \\
\end{array}
\end{align*}
\]

Define

\[
A^-(z) := \max_{i=1, \ldots, n} a_i - \frac{z}{w_i} \quad \text{and} \quad A^+(z) := \min_{i=1, \ldots, n} a_i + \frac{z}{w_i}
\]
Therefore
\[
\min_{x \in \mathbb{R}} \max_{i=1,\ldots,n} w_i |x - a_i| \iff \min_z \text{ s.t. } A^-(z) \leq x \leq A^+(z)
\]

As \( A^-(z) \) and \( A^+(z) \) are monotonically decreasing and increasing, respectively for the \textit{minimal} \( z \), it holds:
\[
A^-(z) = x = A^+(z)
\]

\( \Rightarrow \) There exist \( i \) and \( j \) with \( i, j \in \{1,\ldots,n\} \), so that
\[
a_j - \frac{z}{w_j} = a_i + \frac{z}{w_i} \iff z = \frac{w_i w_j}{w_i + w_j} (a_j - a_i)
\]
1 – Center Problems with $l_\infty$ - Metric on the Line

Algorithm

1. **Compute** for every $i$ and $j$ with $i, j \in \{1, \ldots, n\}$
   \[
   \delta_{ij} = \frac{w_i w_j}{w_i + w_j} (a_j - a_i)
   \]

2. **Determine**
   \[
   \delta_{pq} = \max\{\delta_{ij} : i, j \in \{1, \ldots, n\}\}
   \]

3. **Optimal solution**

   \[
   z^* = \delta_{pq} \quad \text{and} \quad x^* = \frac{w_p a_p + w_q a_q}{w_p + w_q}
   \]

   It holds: $\delta_{ij} = -\delta_{ji}$
   \[\Rightarrow\] it is sufficient to compute $\delta_{ij}$ or $\delta_{ji}$; depending on whether $a_i \leq a_j$ or $a_i > a_j$
1 – Center Problems with $l_\infty$ - Metric on the Line

**Example**

Let $A = \{1, 2, 5, 4, 6\}$ and $w = \{2,3,1,1,2\}$.

**Algorithm**

1. **$i$** | **$j$** | $a_i \leq a_j \ ?$ | $\delta_{ij}$
   --- | --- | --- | ---
   1 | 2 | $1 \leq 2$ | $\delta_{12} = (a_2 - a_1) \cdot (w_1 \cdot w_2) / (w_1 + w_2) = (2 - 1) \cdot 2 \cdot 3 / (2 + 3) = 1.2$
   3 | 1 | $1 \leq 5$ | $\delta_{13} = (a_3 - a_1) \cdot (w_1 \cdot w_3) / (w_1 + w_3) = (5 - 1) \cdot 2 \cdot 1 / (2 + 1) = 2.66$
   4 | 1 | $1 \leq 4$ | $\delta_{14} = (a_4 - a_1) \cdot (w_1 \cdot w_4) / (w_1 + w_4) = (4 - 1) \cdot 2 \cdot 1 / (2 + 1) = 2.0$
   5 | 1 | $1 \leq 6$ | $\delta_{15} = (a_5 - a_1) \cdot (w_1 \cdot w_5) / (w_1 + w_5) = (6 - 1) \cdot 2 \cdot 2 / (2 + 2) = 5.0$
   2 | 3 | $2 \leq 5$ | $\delta_{23} = 2.25$
   4 | 2 | $2 \leq 4$ | $\delta_{24} = 1.5$
   5 | 2 | $2 \leq 6$ | $\delta_{25} = 4.8$
   3 | 4 | $5 \leq 4$ | $\delta_{34} = \delta_{43} = 0.5$
   5 | 5 | $5 \leq 6$ | $\delta_{35} = 0.66$
   4 | 5 | $4 \leq 6$ | $\delta_{45} = 1.33$

2. $\delta_{pq} = \delta_{15} = 5$.

3. $z^* = \delta_{15} = 5$ and $x^* = (w_1 \cdot a_1 + w_5 \cdot a_5) / (w_1 + w_5) = (2 \cdot 1 + 2 \cdot 6) / (2 + 2) = 3.5$
1 – Center Problems with $l_\infty$ - Metric on the Line

Remarks

- In case of identical weights ($w_i = 1, \forall i$) $x^*$ is the center point between the two customers with the smallest and largest coordinate, respectively.

- Set of all points $x$, with weighted distance smaller than or equal to $z^*$ to all customer locations

\[ w_i |x - a_i| \leq z \forall i \iff x \in [A^-(z), A^+(z)] \]

- Complexity of the algorithm: $O(n^2)$. 
1 – Center Problems with $l_\infty$ - Metric

Combination of the solutions of the subproblems

Let $(x_1^*, z_1^*)$ and $(x_2^*, z_2^*)$ be optimal solutions of the two subproblems. Then

$$x^* := (x_1^*, x_2^*)$$

is an optimal solution with

$$z^* = g(x^*) = \max \{g(x_1^*), g(x_2^*)\} = \max \{z_1^*, z_2^*\}$$

Set of all optimal solutions

$$\mathcal{X}^*(g) = \mathcal{X}(g_1) \times \mathcal{X}(g_2)$$

$$= [A_1^-(z^*), A_1^+(z^*)] \times [A_2^-(z^*), A_2^+(z^*)]$$

where $A_1^-(\cdot)$, $A_1^+(\cdot)$ and $A_2^-(\cdot)$, $A_2^+(\cdot)$ are defined for the first and second subproblem respectively.
1 – Center Problems with $l_\infty$ - Metric

Example

Assume that $A = \{(1,4), (2,6), (5,1), (4,2), (6,5)\}$ and $w = \{2,3,1,1,2\}$.

Subproblem 1

The customer locations $A_1 = \{1, 2, 5, 4, 6\}$ are the same as in the previous example

$\Rightarrow \ x_1^* = 3.5$ and $z_1^* = 5$.

Subproblem 2

Customer locations $A_2 = \{4, 6, 1, 2, 5\}$

<table>
<thead>
<tr>
<th></th>
<th>i</th>
<th>j</th>
<th>$a_i \leq a_j$ ?</th>
<th>$\delta_{ij}$</th>
<th></th>
<th>i</th>
<th>j</th>
<th>$a_i \leq a_j$ ?</th>
<th>$\delta_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4 ≤ 6</td>
<td>$\delta_{12} = 2.4$</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>6 ≤ 2</td>
<td>$\delta_{42} = -\delta_{24} = 3.0$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>4 ≤ 1</td>
<td>$\delta_{34} = -\delta_{13} = 2.0$</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6 ≤ 5</td>
<td>$\delta_{52} = -\delta_{25} = 1.2$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>4 ≤ 2</td>
<td>$\delta_{41} = -\delta_{14} = 1.33$</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1 ≤ 2</td>
<td>$\delta_{34} = 0.5$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4</td>
<td>4 ≤ 5</td>
<td>$\delta_{15} = 1.0$</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1 ≤ 5</td>
<td>$\delta_{35} = 2.66$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>6 ≤ 1</td>
<td>$\delta_{32} = -\delta_{23} = 3.75$</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>2 ≤ 5</td>
<td>$\delta_{45} = 2.0$</td>
</tr>
</tbody>
</table>
1 – Center Problems with $l_\infty$ - Metric

2. $\delta_{pq} = \delta_{32} = 3.75$

3. $z_2^* = \delta_{32} = 3.75$ and
   
   $x_2^* = (w_3 \cdot a_{3,2} + w_2 \cdot a_{2,2}) / (w_3 + w_2) = (1 \cdot 1 + 3 \cdot 6) / (1 + 3) = 4.75$

$\Rightarrow x^* = (x_1^*, x_2^*) = (3.5, 4.75)$ is an optimal solution with

$z^* = \max \{z_1^*, z_2^*\} = \max \{5, 3.75\} = 5.$

All optimal solutions:

$x^*(g) = x^*(g_1) \times x^*(g_2)$

$= [A_1^-(z^*), A_1^+(z^*)] \times [A_2^-(z^*), A_2^+(z^*)]$

$= [\max\{-1.5, 0.33, 0, -1, 3.5\}, \min\{3.5, 3.66, 10, 9, 8.5\}] \times$

$[\max\{1.5, 4.33, -4, -3, 2.5\}, \min\{6.5, 7.66, 6, 7, 7.5\}]$

$= \{3.5\} \times [4.33, 6]$
1 – Center Problems with $l_\infty$ - Metric

**Remark**

The **unweighted** $l_1$ – Center Problem corresponds to an **Rhombus-Covering Problem**

**Reason:**

The set $B_1(z) = \{y \in \mathbb{R}^2 : l_1(0, y) = z\}$ describes a rhombus.

**Remark:**

$B_1(z)$ corresponds to an about $45^\circ$ rotated and by the factor $\sqrt{2}$ compressed square.

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Facility Location and Strategic Supply Chain Management

Prof. Dr. Stefan Nickel
1 – Center Problems

The 1 – Center problem with $l_2$ - Metric

Objective function

$$g(x) = \max_{i=1,\ldots,n} w_i l_2(x, a_i)$$

$$= \max_{i=1,\ldots,n} w_i \sqrt{((x_1 - a_{i,1})^2 + (x_2 - a_{i,2})^2)}$$

The unweighted case: The Circle-Covering Problem

Reformulation of the problem

$$\min_{x \in \mathbb{R}^2} \max_{i=1,\ldots,n} l_2(x, a_i) \iff \min_{u.d.N.} \ z \ \ l_2(x, a_i) \leq z \ \ \forall i = 1, \ldots, n$$

$\implies$ **Minimize** $z$, such that the distance from a point $x$ to all customer locations being smaller than or equal to $z$. 
Unweighted 1 – Center Problem with $l_2$ - Metric

**Geometric interpretation of the relation**

\[ l_2(x, a_i) \leq z \quad \forall i = 1, \ldots, n \quad (*) \]

The set of all points with $l_2$ - distance $z$ to the origin

\[ B_2(z) = \{ y \in \mathbb{R}^2 : l_2(0, y) = z \} \]

describes a circle

with center point $0$ and radius $z$. 
Unweighted 1 – Center Problems with $l_2$ - Metric

⇒ If (*) is valid for $(x, z)$, then all customer locations are within a circle with center point $x$ and radius $z$.

The minimal circle-covering problem

Find a circle with minimal radius $z$ and center point $x$ that covers all customer locations, i.e. the distance from $x$ to all customers is smaller than or equal to $z$.

Called: minimal covering circle

A minimal covering circle $(x^*, z^*)$ is also an optimal solution for the unweighted $l_2$ – center problem, and vice versa.
Unweighted 1 – Center Problems with $l_2$ - Metric

**Dual interpretation** of the minimal circle covering problem

Put a circle with **radius** $r$ around each customer location.
Find a **minimal** $r$ such that all circles intersect in **exactly one point** $x$.

⇒ $x$ is the **optimal solution** and $r$ the **optimal objective function value** (radius) of the corresponding 1 – center problem.
Characteristic of the problem

**Minimal covering circle** for all customer locations

Already uniquely determined by the **minimal covering cycle** of just two or three customer locations.

**Notation**

Circle with center point \( x \) and radius \( r \) : \( K(x, r) \)

**Minimal covering circle (MCC) of two customer locations \( a_i \) and \( a_j \)**

The center point \( x \) of the MCC is the center point of the segment between \( a_i \) and \( a_j \), and the radius \( r \) is half of the length of this segment.

\[
x = \left( \frac{a_{i,1} + a_{j,1}}{2}, \frac{a_{i,2} + a_{j,2}}{2} \right) \quad \text{and} \quad r = \frac{1}{2} l_2(a_i, a_j)
\]
Unweighted 1 – Center Problems with $l_2$ - Metric

Minimal covering circle (MCC) of three customers $a_i$, $a_j$, and $a_k \in A$

Case 1:

The triangle $a_i$, $a_j$, and $a_k$ has an obtuse angle ($\geq 90^\circ$)  
$\Rightarrow$ the MCC is defined by the two end vertices of the hypotenuse.

Case 2:

The MCC corresponds to the circumcircle of the triangle $a_i$, $a_j$, and $a_k$.

Circumcircle $UK(x,r)$:

- Center point $x$: Intersection of two perpendicular bisectors of the sides
- Radius $r$: distance from $x$ to one of the three vertices of the triangle.
Unweighted 1 – Center Problem with $l_2$ - Metric

Perpendicular bisector between two points $a_i$ and $a_j$ of the same weight is a straight line

$$MS(a_i, a_j) := m_{ij} \cdot x_1 + b_{ij} := \frac{a_{i,1} - a_{j,1}}{a_{j,2} - a_{i,2}} \cdot x_1 + \frac{a_{j,1}^2 + a_{j,2}^2 - a_{i,1}^2 - a_{i,2}^2}{2(a_{j,2} - a_{i,2})}$$

Intersection point $x$ of two perpendicular bisectors $MS(a_i, a_j) = m_{ij} x_1 + b_{ij}$ and $MS(a_i, a_k) = m_{ik} x_1 + b_{ik}$

$$x = \left( \frac{b_{ik} - b_{ij}}{m_{ij} - m_{ik}}, \frac{m_{ij} \cdot b_{ik} - b_{ij} \cdot m_{ik}}{m_{ij} - m_{ik}} \right)$$

Radius $r = l_2(x, a_i)$
Unweighted 1 – Center Problems with $l_2$ - Metric

**Algorithms** for unweighted $l_2$ – Center Problems

- Enumeration algorithm
- Elzinga-Hearn algorithm
Unweighted 1 – Center Problems with $l_2$ - Metric

Enumeration algorithm

1. Compute all minimal covering circles of two or three customer locations $a_i$, $a_j$ and $a_k \in A$.

2. Delete all of these minimal covering circles which do not cover all customer locations.

3. Determine among the remaining MCCs that one with minimal radius.

Remarks

• A MCC $K(x',z')$ doesn't cover all customer locations if there is a customer $a_i$ for which $l_2(x', a_i) > r'$ applies.

• Computing the MCC for three locations: simplification
  Test, if the MCC that is defined by the two end points of the hypotenuse, also contains the third location.
  If it does, then this already is the MCC of all three locations, i.e. the triangle has an obtuse angle. Otherwise, determine the MCC by using the formulas of case 2.
Unweighted 1 – Center Problems with $l_2$ - Metric

Example

Let $A = \{(1,1), (2,5), (3,3), (4,2)\}$ and $w = \{1,1,1,1\}$.

Method

1. Points | MCC: $K(x,z)$ | Covering ? | Points | MCC: $K(x,z)$ | Covering ?
--- | --- | --- | --- | --- | ---
$a_1, a_2$ | (1.5, 3), 2.06 | $l_2(x, a_4) > r$ | $a_3, a_4$ | (3.5, 2.5), 0.71 | $l_2(x, a_1) > r$
$a_1, a_3$ | (2, 2), 1.41 | $l_2(x, a_2) > r$ | $a_1, a_2, a_3$ | $K(a_1,a_2)$ | -
$a_1, a_4$ | (2.5, 1.5), 1.58 | $l_2(x, a_2) > r$ | $a_1, a_2, a_4$ | (2.05, 2.86), 2.14 | Ja !
$a_2, a_3$ | (2.5, 4), 1.12 | $l_2(x, a_1) > r$ | $a_1, a_3, a_4$ | $K(a_1,a_4)$ | -
$a_2, a_4$ | (3, 3.5), 1.80 | $l_2(x, a_1) > r$ | $a_2, a_3, a_4$ | $K(a_2,a_4)$ | -

2. Only the minimal covering circle of the customer locations $a_1, a_2$ and $a_4$ covers all customers.

3. The MCC of the customer locations $a_1, a_2$ and $a_4$ is minimal for the whole problem.
Remark

All in all, there are \( \binom{n}{2} = O(n^2) \) pairs and \( \binom{n}{3} = O(n^3) \) triples of customer locations

\( \Rightarrow \) Complexity of the algorithm: \( O(n^4) \)

Problem

For hundreds of customer locations this takes too much time!

Idea

Construct a sequence of minimal covering circles with continuously increasing radius.

Reminder: Characteristic of MCCs

The minimal covering circle of a set of customer locations is uniquely determined by two or three locations.
Unweighted 1 – Center Problems with $l_2$ - Metric

Elzinga & Hearn Method

1. Start with the MCC of two arbitrary customer locations $a$ and $b$.
2. If all customers are covered by the circle $\Rightarrow$ Step 9.
3. Choose a customer $c$ who is not covered yet and construct the MCC for these three customers.
4. If all customers are covered by the circle $\Rightarrow$ Step 9.
5. If the MCC is determined by two customers $a'$ and $b'$
   $\Rightarrow$ continue with $a'$ and $b'$ with step 3., otherwise
   $\Rightarrow$ continue with all three of them with step 6.
6. Choose another customer $d$ who is not covered yet and construct (with the aid of the Enumeration algorithm) the MCC for these four customers.
7. If all customers are covered by the MCC $\Rightarrow$ Step 9.
8. If the MCC is determined by two customers $a''$ and $b''$
   $\Rightarrow$ continue with $a''$ and $b''$ with step 3., otherwise
   $\Rightarrow$ continue with three MCC-defining customers with step 6.
9. The current MCC covers all customer locations and is therefore optimal.
Unweighted 1 – Center Problems with $l_2$ - Metric

Example

Let again $A = \{(1,1), (2,5), (3,3), (4,2)\}$ and $w = \{1,1,1,1\}$.

Method

1. Start with the customers $a_1$ and $a_3$. MCC = $K((2, 2), 1.41)$
2. $a_2$ and $a_4$ not covered
3. Choose $a_4$. MCC of customers $a_1$, $a_3$ and $a_4 = K((2.5, 1.5), 1.58) = K(a_1,a_4)$. 
4. $a_2$ not covered
5. The MCC is determined by $a_1$ and $a_4$.
3. Choose $a_2$. MCC of customers $a_1$, $a_2$ and $a_4 = K((2.05, 2.86), 2.14)$. 
4. All customers covered!
9. $K((2.05, 2.86), 2.14)$ is the minimal covering cycle for all locations.

$\Rightarrow x = (2.05, 2.86)$ is the optimal location for the $l_2$– center problem and $r = 2.14$ the optimal objective function value.
Unweighted 1 – Center Problems with $l_2$ - Metric

Remark

The radius of the constructed MCCs increases with each iteration.

The weighted case

The Enumeration algorithm and the Elzinga & Hearn Method can both be extended to the weighted case.

However the computation of minimal covering circles for two and three customer locations becomes more complicated.

Because the perpendicular bisector of the side of two differently weighted points is a circle.